JOURNAL OF COMPUTATIONAL PHYSICS 89, 301–318 (1990)

# A Particle-Grid Superposition Method for the Navier–Stokes Equations*

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Received February 27, 1989; revised July 11, 1989

This paper presents a particle-grid superposition method for solving the Navier–Stokes equations. Starting from a deterministic vortex method, the idea is to superpose a finite difference method where large distortions of the particle distribution occur. A matching between both methods is proposed which eventually leads to a domain decomposition strategy that makes it possible to simulate viscous flows at high Reynolds numbers with a limited number of points. Numerical illustrations of the method are given for flows past a cylinder at Reynolds numbers of 3000 and 9500. 

## 1. INTRODUCTION

Viscous flows at high Reynolds numbers are a challenging problem for applied mathematicians. Many questions related to the limiting process when the Reynolds number tends to infinity remain unanswered, and numerical methods are now attractive alternatives to investigate those problems. However, the resolution of the very small scales that are present for large Reynolds numbers result for conventional numerical methods in memory requirements that have so far prevented a direct approach for turbulence, even with supercomputers.

From this point of view, vortex methods seem to be a promising approach. Two features, in particular, of these methods make it possible to hope to reach high Reynolds numbers with an affordable number of points: these methods have been primarily designed for inviscid flows and therefore perform well in convection dominated problems; moreover, only the support of the initial vorticity needs to be discretized, and in most situations the size of this region decreases when the Reynolds number increases. For instance, in the case of a flow past an obstacle the number of points can be expected to be $O(\sqrt{\mathrm{Re}})$, as compared to the $O(\mathrm{Re})$ number of points required for a finite difference method. For the validity of a particle method, the underlying assumption is that, given a proper particle

* Research supported by ONR Contracts N00014-86-K-0691, N00014-88-K-0414 and NSF Grant DMS88-11863.



0021-9991/90 $3.00

discretization of both the initial vorticity and the source terms at the boundaries, the particles moving with the fluid will maintain the desired resolution. A popular particle method for solving viscous flows is the random walk method [3]. This method has been successfully applied to a number of situations. However, its accuracy is questionnable and it may be desirable to derive different treatments of the diffusion, while keeping the nice features of particle methods for convection dominated flows. An attempt in that direction are the so-called deterministic vortex methods [10]. These methods based on integral approximations of diffusion operators enjoy nice convergence results and have recently proved to be a potentially accurate alternative to random walk methods [4]. However, their accuracy is obviously conditioned by the ability of the moving grid made up by the particles to represent the smallest scales. Unfortunately in non-smooth flows or where high gradients of the velocity are present, particles are most likely to accumulate in certain zones while missing in other locations. In the later case the accuracy of a diffusion process involving only the particles is doubtful. Moreover, stability problems also occur that are related to local accumulations of particles, requiring very small time steps. A possible way to deal with these problems would be to remesh the computational domain, but, in addition to leading to complex codes, this process might deteriorate the advantages of the self adaptativity of the vortex methods in the treatment of the convection.

The strategy chosen here to overcome accuracy problems in regions where particle are missing is to superpose a fixed grid whose nodes are used together with Lagrangian particles to resolve the diffusion, thereby enforcing a minimal resolution everywhere. The grid points can be considered as non moving particles; therefore the diffusion formula involved in the deterministic vortex methods may be applied to these points as well, provided their volumes are computed in a consistent way. The computation of these volumes, which can be viewed as quadrature weights associated to the locations of the particles, is actually the key point to allow a nice transfer of the vorticity between the grid and the lagrangian particles. Once it is done, the resulting algorithm can be considered as a correction of the original vortex method. For the method to be economic, the underlying assumption is that the corrections occur only in very specific zones, which we believe is reasonable in many situations where these zones can be determined from physical grounds. Let us point out that for the resolution of the diffusion, the vorticity is never involved in any assignment step, which makes this method definitely different from the conventional particle-in-cell methods. Also, as it will be seen in Section 5, it significantly differs from a hybrid method proposed in [11].

An outline of this paper is as follows: in Section 2 we recall the definitions of deterministic vortex methods for the Navier–Stokes equations. We discuss stability problems and propose a modified method which overcomes these difficulties. In Section 3 we present the particle-grid superposition method. In Section 4 we describe a vortex-in-cell algorithm that involves these particle-grid superposition techniques for solving 2D viscous flows in exterior domains and we show preliminary numerical results for flows past a cylinder at Reynolds numbers of 3000

and 9500, for which experimental results are available. A comparison between our results and other numerical results for the same Reynolds numbers as given in [9] also allows us to highlight both the possibilities and the limits of our method. Finally in Section 5 we interpret our superposition techniques in terms of domain decomposition terminology, we draw some conclusions, and we contrast our method with the one in [11].

## 2. Deterministic Vortex Methods for Navier–Stokes Equations

Let us consider the vorticity formulation of the two-dimensional Navier–Stokes equations in the whole plane:

$$\frac{\partial \omega}{\partial t} + \nabla(u \otimes \omega) - \nu \Delta \omega = 0 \tag{1}$$

$$\omega(\cdot, 0) = \omega_0 \tag{2}$$

$$\operatorname{curl} u = \omega \tag{3}$$

$$\operatorname{div} u = 0 \tag{4}$$

$$u \overset{\infty}{\to} u_\infty . \tag{5}$$

Particle methods consist in approximating the vorticity by a set of particles determined by their weights $\alpha_p$ and locations $x_p$. A weight $\alpha_p$ combines the local value of the vorticity $\omega_p$ and the volume $v_p$ of the particle, according to the formula $\alpha_p = v_p \omega_p$. The particles move as material points which means that the $x_p$ satisfy

$$\frac{dx_p}{dt} = u(x_p(t), t); \qquad x_p(0) = x_p^0. \tag{6}$$

In the incompressible case the volumes of the particles are constant in time and correspond to the proper weights associated to the quadrature points $x_p^0$. Deterministic vortex methods consist in accounting for the diffusion by making the weights of the particles evolve according to the equations:

$$\frac{d\omega_p}{dt} = \frac{\nu}{\eta^2} \sum_{p'} (\omega_{p'} - \omega_p)\, v_{p'} \Lambda_\eta(x_p - x_{p'}); \qquad \omega_p(0) = \omega(x_p^0, 0). \tag{7}$$

In the above equations $x_p^0$ and $\alpha_p^0$ are the initial positions and weights of the particles, $\eta$ is a small parameter, and

$$\Lambda_\eta(x) = \eta^{-2} \Lambda\left(\frac{x}{\eta}\right),$$

where $A$ is a symmetric kernel such that

$$\int x_i^2 A(x)\,dx = 2, \qquad i = 1, 2. \tag{8}$$

For the convergence of such methods we refer to [10] for the linear case, and to [8] for the Navier–Stokes equations. Let us just mention that this class of methods is based on integral approximations of the diffusion operator followed by a quadrature of the integral along particles, and that the order of the resulting approximation is primarily based on the moment properties of the kernel $A$. For instance, only under the above assumptions on $A$, a Taylor expansion of $\omega$ at the point $x$ easily yields

$$A\omega(x) = \frac{1}{\eta^2}\int (\omega(y) - \omega(x))\, A_\eta(x - y)\,dy + O(\eta^2).$$

The formula (7) then derives from a quadrature of this integral evaluated at the particle $x_p$, with quadrature points $x_{p'}$ and quadrature weights $v_{p'}$. If in addition the moments of order 4 of the kernel $A$ vanish, the resulting integral approximation is of order 4 (we will refer in the sequel to such kernels as kernels of order 4).

We now illustrate some important features of these diffusion techniques. For simplicity we focus on the one-dimensional version of (7) for a very simple kernel shape,

$$A(x) = \begin{cases} 12(1 - |x|) & \text{if } |x| \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

We also assume that we are starting with a uniform distribution of particles, with a mesh size $h$ (that is $v_p \equiv h$), and that the density of particles remain locally constant at all times. In other words $\mathbf{R}$ can be divided at all times into subdomains where the distance between particles is constant:

$$\mathbf{R} = \bigcup_k \Omega_k; \qquad \begin{cases} h_k = \text{distances between nearest particles in } \Omega_k \\ N_k = \eta/h_k. \end{cases}$$

In $\Omega_k$ the right-hand side of the diffusion scheme (7) actually reduces to a finite difference scheme on a grid with grid size $h_k$, the stencil of which can be easily checked to be

$$12\frac{v}{\eta^2}\frac{h}{\eta}\Big[\ldots, 0, \underbrace{\frac{1}{N_k+1}, \ldots, \frac{N_k}{N_k+1}, -N_k, \frac{N_k}{N_k+1}, \ldots, \frac{1}{N_k+1}}_{2N_k+1}, 0, \ldots\Big]. \tag{9}$$

This stencil is readily seen to be consistent with

$$12\frac{v}{\eta^2}\frac{h}{\eta}\frac{h_k^2}{N_k^2}\left[S_2^{N_k}-\frac{S_3^{N_k}}{N_k+1}\right]\frac{d^2}{dx^2},$$

where $S_p^n=\sum_{i=0}^{i=n} i^p$.

Given the explicit values of $S_2^n$ and $S_3^n$,

$$S_2^n=\tfrac{1}{6}n(2n+1)(n+1); \qquad S_3^n=\tfrac{1}{4}n^2(n+1)^2,$$

we observe that $S_2^{N_k}-S_3^{N_k}/(N_k+1)\simeq N_k^3/12$ when $N_k\to\infty$. This allows to check that if $N_k\simeq\bar{N}=\eta/h$ (that is if the density remains as initially) the diffusion formula (7) is consistent with the original diffusion equation in the limit $\bar{N}\to\infty$.

We now come to a stability analysis, which turns out to be very simple and enlightening in the present model situation. Assuming that we use a forward Euler scheme for discretizing (7), the resulting algorithm can be written

$$[\omega(t+\Delta t)]=[\omega(t)]+12\frac{v\Delta t}{\eta^2}\frac{h}{\eta}[A][\omega(t)],$$

where $A$ is a matrix whose block corresponding to $\Omega_k$ have rows given between brackets in (9). As a result, the maximum eigenvalue of A is $\max_k N_k$ and therefore the scheme (7) is stable under the condition:

$$12\frac{v\Delta t}{\eta^2}\frac{N_k}{\bar{N}}\leqslant 2. \tag{10}$$

This means that if the density remains constant ($N_k\simeq N$) the stability condition looks like classical stability conditions for finite difference methods:

$$\frac{v\Delta t}{\eta^2}\leqslant C.$$

However, if particles locally accumulate then $N_k$ may become much larger than $\bar{N}$, resulting in drastic restrictions on the time step. A simple way to overcome these difficulties is to compute a local value for the parameter $\eta$, which is an approximation of the actual value of $\eta$ when the particle distribution is smooth and which corrects the effects of the accumulations of particles.

A possible formula is

$$\eta_p=\lambda^{-1}h\sum_{p'}\Lambda\left(\frac{x_p-x_{p'}}{\eta}\right), \qquad \text{where} \quad \lambda=\int\Lambda(x)\,dx.$$

If we replace $\Lambda_\eta$ by $(1/\eta_p)\,\Lambda(x/\eta)$, we can check the stability on our 1D example

under a condition that is independent of $N_k$: in this case the above formula leads to

$$\eta_p = \lambda^{-1}\lambda \frac{h}{h_k} = \frac{h}{h_k} \qquad \text{for} \quad x_p \in \Omega_k,$$

and the stability condition (10) becomes

$$12 \frac{\nu \Delta t}{\eta^2} \frac{h}{\eta_p} N_k = 12 \frac{\nu \Delta t}{\eta^2} \leqslant 2.$$

The time step can now be choosen independent of $N_k$. In 2D the natural extension of the proposed correction is to set

$$\eta_p^2 = \lambda^{-1} \sum_{p'} v_{p'} \Lambda\left(\frac{x_p - x_{p'}}{\eta}\right), \qquad \Lambda_{\eta, p} = \frac{1}{\eta_p^2} \Lambda\left(\frac{x}{\eta}\right).$$

The modified diffusion formula is then:

$$\frac{d\omega_p}{dt} = \frac{\nu}{\eta^2} \sum_{p'} (\omega_{p'} - \omega_p)\, v_{p'} \Lambda_{\eta, p}(x_p - x_{p'}); \qquad \omega_p(0) = \omega(x_p^0, 0). \tag{11}$$

The two-dimensional case raises the following question: in a 2D incompressible flow the mechanism for the accumulations of particles is still one-dimensional and accumulations of particles along one direction usually go with stretching along other directions (a typical example of such a situation is in the vicinity of a stagnation point). In this case the correction in the coefficient $\eta_p$ is likely to "penalize" the diffusion indifferently along all directions, which is not desirable. To overcome this difficulty, it is possible, still on the basis of our one-dimensional analysis, to incorporate into our diffusion scheme a correction method which distinguishes between the two axis directions (that is, which computes two values, $\eta_{p,1}$, $\eta_{p,2}$, instead of one). However, the overall method has not shown a significant sensitivity to such a distinction and for simplicity we will omit its description here.

Let us finish this section with an unexpected (but of practical interest) consequence of the corrections proposed here in the case where boundaries are present. Let us go back to our one-dimensional model problem and consider a particle located at the boundary. In this case the formula for $\eta_p$ becomes

$$\eta_p = h/2h_k$$

and the stencil at the particle $x_p$ is now

$$24 \frac{\nu}{\eta^2} h_k \left[ -\frac{N_k}{2}, \frac{N_k}{N_k + 1}, \ldots, \frac{1}{N_k + 1}, 0, \ldots \right].$$

It is then readily seen that this stencil yields a value of the diffusion at the particle $x_p$ which coincides with what would have been found if particles had been created on the other side of the boundary, with equal vorticity, before applying formula (9) in the whole space. This means that the boundary condition implicitly implemented in the scheme is a homogeneous Neumann boundary condition. The implication of this observation for the two-dimensional case will be discussed in Section 4.

## 3. The Particle-Grid Superposition Method

Even in its modified form (11), the diffusion scheme can obviously not accurately handle regions where particles are missing. More precisely, the consistency of the diffusion method is primarily based on the moment properties (8) of the kernel $\Lambda$, and on the accuracy of the following approximations:

$$2\eta^2 = \int (x_p - x_{p'})_i^2 \, \Lambda_\eta(x_p - x_{p'}) \, dx_{p'}$$

$$\simeq \sum_{p'} v_{p'}(x_p - x_{p'})_i^2 \, \Lambda_\eta(x_p - x_{p'}), \qquad i = 1, 2.$$

This accuracy is very much conditioned by the smoothness of the flow map which determines the positions of the particles at every time. For high Reynolds numbers, the vorticity can reach high values in certain regions, and so do the derivatives of the flow map. As a result, one can expect a dramatic deterioration of the accuracy of the particle formula for the diffusion in these regions. This is particularly unpleasant since in these regions of high gradients the flow is likely to be diffusion dominated rather than convection dominated. Assuming that their locations can be predicted from physical grounds (the very neighbourhood of the obstacle in our case), the remedy consists in superposing a grid on the particle distribution in order to enforce a minimal resolution for the diffusion there. For simplicity we wish to consider the grid points as non-moving particles and to incorporate them in the diffusion formula (11). For that we need to assign volumes to these points. We indicate two ways of achieving that.

In the sequel the symbols $y$ and $w$, with indices $i, j$, refer to grid points (locations and volumes), while the symbols $x, v, p, p'$ refer to the lagrangian particles. In light of the preceding remarks, a desirable property for the new particle distribution (including grid points) would be that, at each grid point $y_i$,

$$\sum_{p'} v_{p'}(y_i - x_{p'})^2 \, \Lambda_\eta(y_i - x_{p'}) + \sum_j w_j(y_i - y_j)^2 \, \Lambda_\eta(y_i - y_j) = 4\eta^2. \tag{12}$$

Introducing $w_j$ as new degrees of freedom allows to ask (12) to be satisfied at each point $y_i$. This results in a linear system with unknown vector $w = (w_j)_j$ which can be rewritten as

$$\sum_j w_j(y_i - y_j)^2 \Lambda_\eta(y_i - y_j) = 4\eta^2 - \sum_p v_p(y_i - x_p)^2 \Lambda_\eta(y_i - x_p) \qquad \text{for all } i. \quad (13)$$

If we return to the one-dimensional example in Section 2, the above requirement can be shown to make the algorithm reduce *exactly* to a conventional finite difference scheme at the points $y_i$. Let us now assume that the grid size is $\eta/2$. If $M$ is the matrix with general term $a_{ij} = (y_i - y_j)^2 \Lambda_\eta(y_i - y_j)$, we next observe that $M$ is close to $16I$ (where $I$ denotes the identity matrix), for we can write

$$[Mw]_i = \frac{4}{\eta^2} \sum_j \left(\frac{\eta}{2}\right)^2 w_j a_{ij}.$$

Therefore $(\eta^2/4)[Mw]_i$ can be interpreted as a discrete convolution of $w$ by the kernel $x^2 \Lambda_\eta(x)$ at the point $y_i$. Since, by (8), the integral of this kernel is $4\eta^2$, neglecting terms of higher order gives

$$[Mw]_i \simeq \frac{4}{\eta^2} w_i 4\eta^2 = 16 w_i.$$

As a result, a natural way to solve the linear system (13) is to construct an iterative scheme on the decomposition $M = 16I + (M - 16I)$. This yields the formula

$$w_i^{n+1} = \frac{\eta^2}{4} + w_i^n - \frac{1}{16}\left\{\sum_p v_p(x_p - y_i)^2 \Lambda_\eta(x_p - y_i) + \sum_j w_j^n (y_j - y_i)^2 \Lambda_\eta(y_j - y_i)\right\}. \quad (14)$$

Let us comment now on the correction induced by the resolution of (12). Consider first the case when the original distribution of particles $x_p$ is "ideal." In this case, quadratures along these particles are accurate and, in particular, for all $y_i$,

$$4\eta^2 \simeq \sum_p v_p(x_p - y_i)^2 \Lambda_\eta(x_p - y_i).$$

Therefore solving (13) yields $w_i \simeq 0$, which means that in the diffusion formula (11) the grid does not affect the vorticity on the particles $x_p$. This is reasonable, since those particles $x_p$ should be enough by themselves for solving the diffusion. On the other hand, if lagrangian particles are missing in some domain $\Omega_k$, then for $y_i$ in $\Omega_k$ we must have

$$\sum_j w_j(y_j - y_i)^2 \Lambda_\eta(y_j - y_i) \simeq 4\eta^2.$$

Then, as already noticed, the diffusion scheme reduces to a standard finite difference scheme on the grid $(y_j)$.

Between these two extreme cases, the resolution of (13) ensures a matching between the grid and the lagrangian particles, and we will discuss this point of view when we interpret the method within the framework of domain decomposition techniques in Section 5.

Another way to compute the volumes $w_j$ would be to consider any basis function $\phi_\eta$ associated with the grid $(y_i)$ and to use the fact that computing the volumes of the cells of the grid, using either the integral of $\phi_\eta$ or the quadrature of this integral along particles of both species $y_i$ and $x_p$, give the same result; that is,

$$\sum_j w_j \phi_\eta(y_j - y_i) + \sum_p v_p \phi_\eta(y_i - x_p) = \int \phi_\eta(y_i - x)\, dx = \eta^2 \qquad \text{for all } i. \tag{15}$$

This formula leads once again to inverting a band system whose band width depends on the size of the support of $\phi_\eta$. Choosing for instance $\phi$ to be a tensor product of characteristic functions results in a diagonal system whose solution is

$$w_i = 1 - \eta^{-2}\left(\sum_p v_p \phi_\eta(y_i - x_p)\right) \qquad \text{for all } i. \tag{16}$$

In our calculations we have actually used (16) as a way to compute the first values $w_j^0$ to be inserted in the right-hand side of (14). Then one iteration of the iterative process (14) has been performed. The final diffusion scheme, accounting also for the modifications described in Section 2, is

$$\begin{aligned}
\frac{d\omega_i}{dt} &= \frac{\nu}{\eta^2} \sum_{p',j} (\omega_j - \omega_i)\, w_j \Lambda_{\eta,i}(y_i - y_j) \\
&\quad + (\omega_{p'} - \omega_i)\, v_{p'} \Lambda_{\eta,i}(y_i - x_{p'}) \\
\frac{d\omega_p}{dt} &= \frac{\nu}{\eta^2} \sum_{p',j} (\omega_j - \omega_p)\, w_j \Lambda_{\eta,p}(x_p - y_j) \\
&\quad + (\omega_{p'} - \omega_p)\, v_{p'} \Lambda_{\eta,p}(x_p - x_{p'}),
\end{aligned} \tag{17}$$

with initial conditions $\omega_i(0) = \omega(y_i, 0)$, $\omega_p(0) = \omega(x_p^0, 0)$.

Besides maintaining a minimal resolution, it has been observed that another advantage of the approach just described is to guarantee a certain isotropy in the particle distribution. This may be desirable in particular in the neighbourhood of the boundary and in the context of vorticity creation algorithms. In particular, implementing Chorin's algorithm (in order to satisfy the no slip condition) with its optimal consistency requires us to solve the transport diffusion equation with homogeneous boundary condition (see [6] for a discussion of the later point). It has been noticed in the 1D model problem that the particle diffusion scheme, as modified in Section 2, precisely accounts for this boundary condition. As for the 2D case, it is not difficult to realize that the same property will be true, but only when the distribution of particles is isotropic near the boundary. This property may be

violated by the lagrangian particles, once again due to the distortions in the flow, and introducing non moving particles with proper volumes restores the missing isotropy. We believe that this is the reason why it has been found satisfactory to implement the particle-grid superposition method along with the Chorin's vorticity creation algorithm.

## 4. Numerical Experiments

We now describe a vortex-in-cell algorithm which uses the idea described above for solving viscous flows at high Reynolds numbers in exterior domains. Let $\Omega^0$ be a bounded domain, with boundary $\Gamma$, and $\Omega$ the complementary of $\Omega^0$. In our experiments $\Omega^0$ will be a disc of radius 1 and the velocity at infinity will be taken uniform. On the obstacle, we assume that the flow satisfies the no-slip condition:

$$u = 0 \qquad \text{on } \Gamma. \tag{18}$$

In this case the system (1)–(5) must be supplemented with two boundary conditions:

$$u \cdot n = 0 \qquad \text{on } \Gamma \tag{19}$$

$$\frac{\partial \omega}{\partial n} = g \qquad \text{on } \Gamma, \tag{20}$$

where the function $g$ has to be determined. The derivation of $g$ and how this boundary condition leads to a vorticity creation algorithm are delicate matters (see [1, 6]; [6] in particular describes a vorticity creation algorithm which is compatible with grid-free deterministic particle methods). However, as pointed out above, the superposition of a grid for the resolution of the diffusion restores the isotropy of the particle-grid mesh. As a result it has been found satisfactory to implement in this context Chorin's vorticity creation algorithm: a layer of fixed particles is considered in the neighbourhood of $\Gamma$ with a mesh size $\delta l$, and at each time step those particles are charged with an amount of vorticity equal to twice that of the local slip of the velocity. This layer of particles actually coincides with the first layer of grid points incorporated in the diffusion scheme. In our experiments the slip on the surface of the cylinder, measured by its $L^2$ norm is of the order of 2%.

The resolution of the diffusion equation (11) has proved to be sensitive to the shape of the kernel $A$ and it has been fruitful to choose a fourth-order kernel, as given in [4],

$$A(x) = \frac{4}{\pi} \frac{18 - 6|x|^2}{(1 + |x|^2)^5}.$$

The value of $\eta$ has been choosen equal to $2\sqrt{\nu}$ (given the fact that we were interested in high Reynolds numbers, the viscosity is here considered as a perturbation parameter; we refer to [10] for the mathematical aspects of this point of view). In order to avoid a $O(N^2)$ cost for the diffusion formula (17) we actually drop particles which are far enough. For that we cover $\Omega$ with boxes with size $c\eta$. Each particle is then given a box number and, reciprocially, particles within a given box are linked to the original list. In the formula (17), only those particles which are in a box connected to the box containing $y_i$ (or $x_p$) are called. The resulting cost for the implementation of (17) is $O(N)$. The truncation error involved in this process can be easily estimated by comparing the integrals in (8), with the integrals of the same quantities over a box of diameter $c$. In our calculations the value $c = 0.75$ has been found satisfactory.

For the resolution of the elliptic system (3)–(5), a vortex-in-cell methodology has been chosen. Actually this choice, which is clearly non-essential in the overall algorithm, has been made for the sake of simplicity; in particular, VIC methods easily handle boundary conditions of the type (19) and are cheap. Moreover, recent results (see [7]) show that it is possible to design processing techniques which make VIC methods accurate. The particular VIC scheme used is the one described in [5]. It consists in a TSC scheme for the assignment (that is the vorticity carried by a particle is shared by the three closest grid points in both directions), along with a piecewise linear interpolation scheme for the computation of the velocities at the particles and a second-order Poisson solver. Contrasting with the above observation concerning the fourth-order resolution of the diffusion, using a fourth-order solver for the Laplace equation has not proved to change significantly the results. A polar grid has been used in the domain $1 \leqslant r \leqslant 5.5$; $0 \leqslant \theta \leqslant 2\pi$. The artificial boundary condition for $r = 5.5$ is simply $u = u_\infty$.

For the time discretization of the various ordinary differential equations we have choosen a forward Euler scheme with a time step $\Delta t = 0.01$. An outline of the algorithm is as follows:

- Resolution of the convection-diffusion equation
  1. VIC resolution of (3)–(5) and resolution of (6): $(x_p^n)_p \to (x_p^{n+1})_p$.
  2. Resolution of (17): $(\omega_{i,p}^n)_{i,p} \to (\omega_{i,p}^{n+1})_{i,p}$.

- Vorticity creation at the boundary: computation of the slip at the boundary and charge of the first layer of particles with the corresponding vorticity.

- Correction of the volumes $(w_j)_j$ for the layers of grid points
  1. Resolution of (16).
  2. Resolution of (14).

We have performed experiments corresponding to Reynolds numbers of 3000 and 9500 ($\mathrm{Re} = 2/\nu u_\infty$). A constant number of layers of particles (15) and grid points (10) were initialized in the boundary layer. The width $\rho$ of this region was

computed according to the formula $\rho = 5\sqrt{2\pi/\mathrm{Re}}$. Then particles were radially generated upstream in the boundary layer to maintain there a resolution proportional to the square root of the viscosity in the tangential direction. Actually, to avoid useless accumulations of particles at the upstream stagnation point, lagrangian particles were only generated in the region $-7\pi/8 \leqslant \theta \leqslant 7\pi/8$. It must be pointed out that no a priori symmetry has been imposed for the numerical solution. For $\mathrm{Re} = 3000$ the resulting number of points (particles of both species) ranged between 4700 (for $t = 0$) and 6900 (for $t = 4$), while for $\mathrm{Re} = 9500$ the corresponding numbers of points were 7800 and 11500. The increase between the two values of Re results only from a refinement in the tangential direction. For the resolution of the elliptic system, the same very rough $150 \times 200$ polar grid has been used.

Figure 1 shows the initial distribution of particles of both species. In Fig. 2 the lagrangian particles distribution downstream around the cylinder is presented at time 2.5 for $\mathrm{Re} = 9500$, showing holes as well as accumulations of particles along the lines. As a numerical measure of these distortions, in Fig. 3 we have plotted the evolution in time of the two following quantities: on the one hand, the maximum number of particles within one of the boxes of size $\eta$ introduced for the implementation of the diffusion formula divided by a factor 10 and on the other hand, the maximum value of the ratio $\eta_p^2/\eta^2$. It can be seen that, as particles accumulate in certain boxes, the latter number, which was originally close to 1, can be multiplied by a factor 7, justifying that locally the original diffusion formula could lead to stability problems. Figure 4 presents the flow structure at time 5, for $\mathrm{Re} = 3000$. In

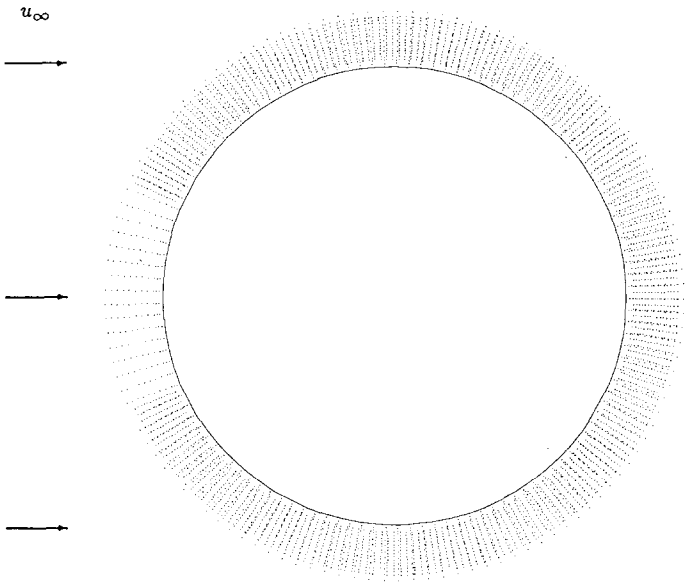


FIG. 1. Grid-particle initialization for $\mathrm{Re} = 3000$.

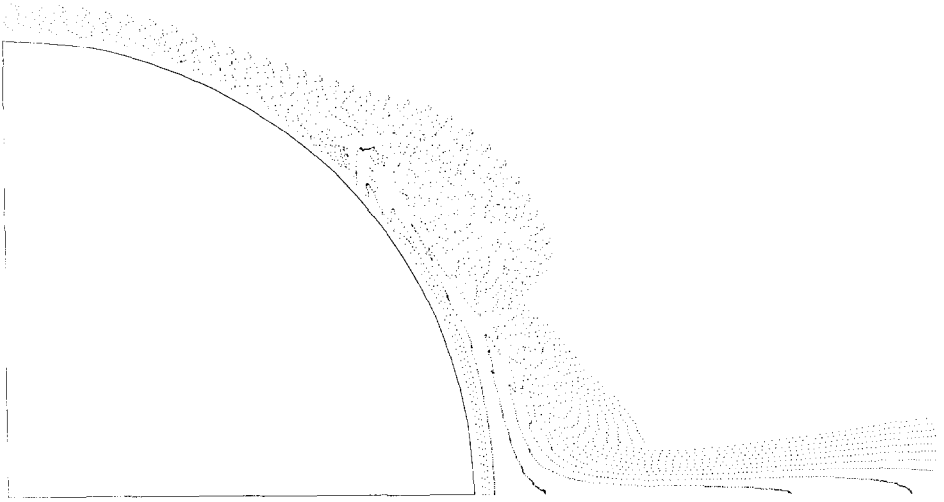

FIG. 2. Repartition of the lagrangian particles behind the cylinder at $t = 2.5$ for $\mathrm{Re} = 9500$.

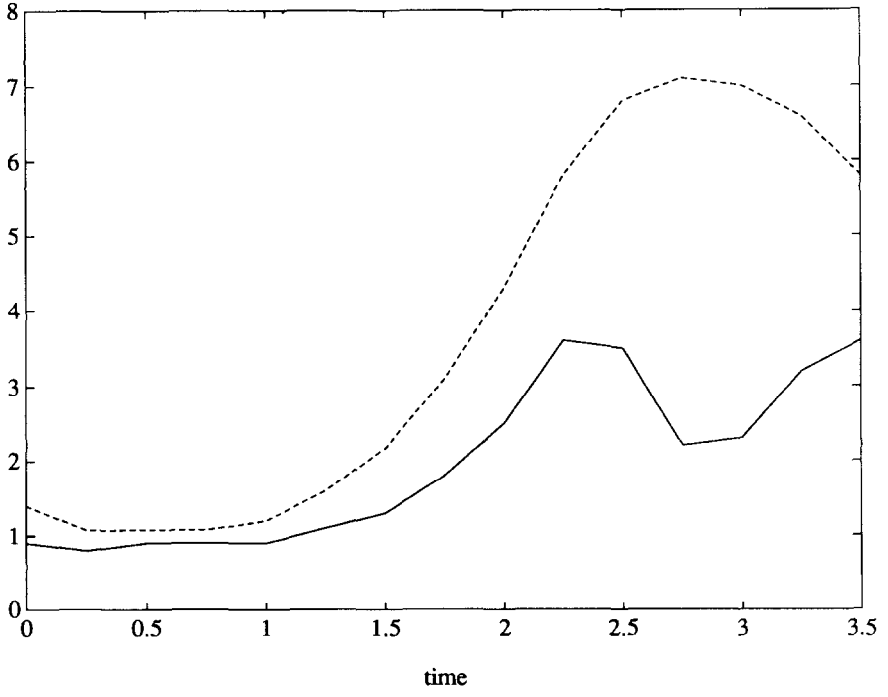


FIG. 3. Maximum value of $\eta_p^2/\eta^2$ (dotted line) and fill-up of the $\eta$-boxes for $\mathrm{Re} = 9500$.

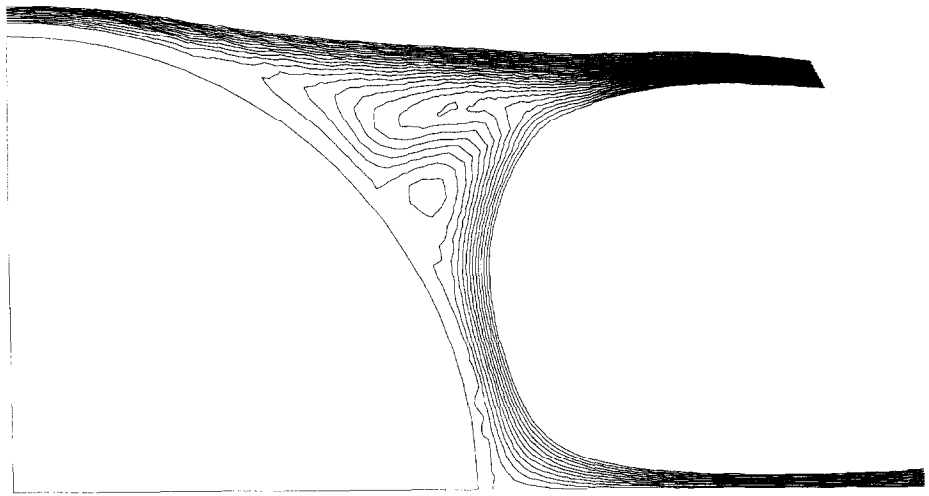

FIG. 4. Flow structure at $t = 5$ for $\mathrm{Re} = 3000$.

the forewake, the presence of two secondary eddies is visible. Figures 5 and 6 give the flow structure at $t = 4.0$ and $t = 4.5$ for $\mathrm{Re} = 9500$. In Fig. 7 we have plotted the values of the vorticity on the surface of the cylinder at $t = 4.0$ for both Reynolds numbers. Some qualitative features of the modifications in the flow can be recognized on this picture: presence of the secondary eddies attested to by a rapid change of the sign in the vorticity, and the fact that the maximum vorticity is

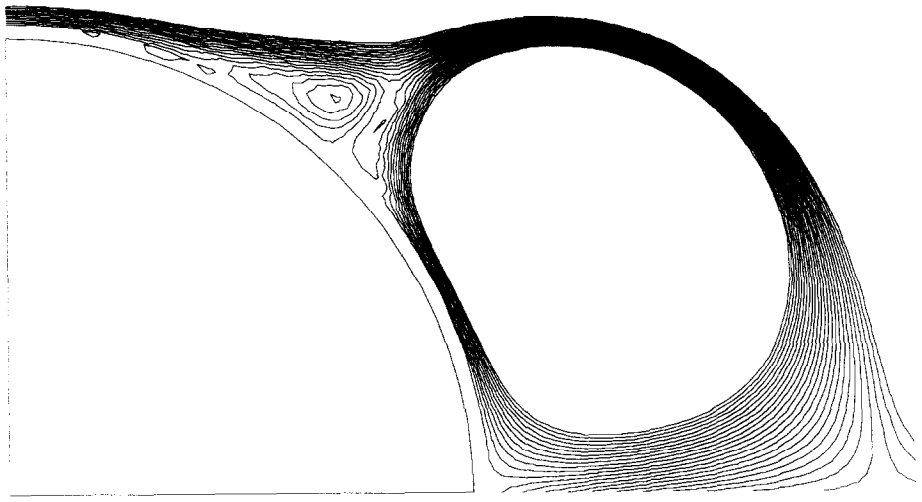

FIG. 5. Flow structure at $t = 4$ for $\mathrm{Re} = 9500$.

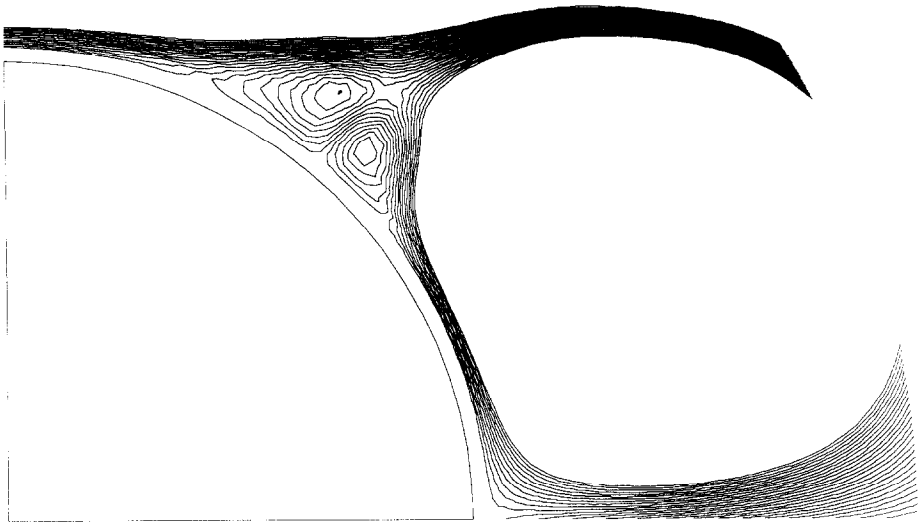

FIG. 6. Flow structure at $t = 4.5$ for $\mathrm{Re} = 9500$.

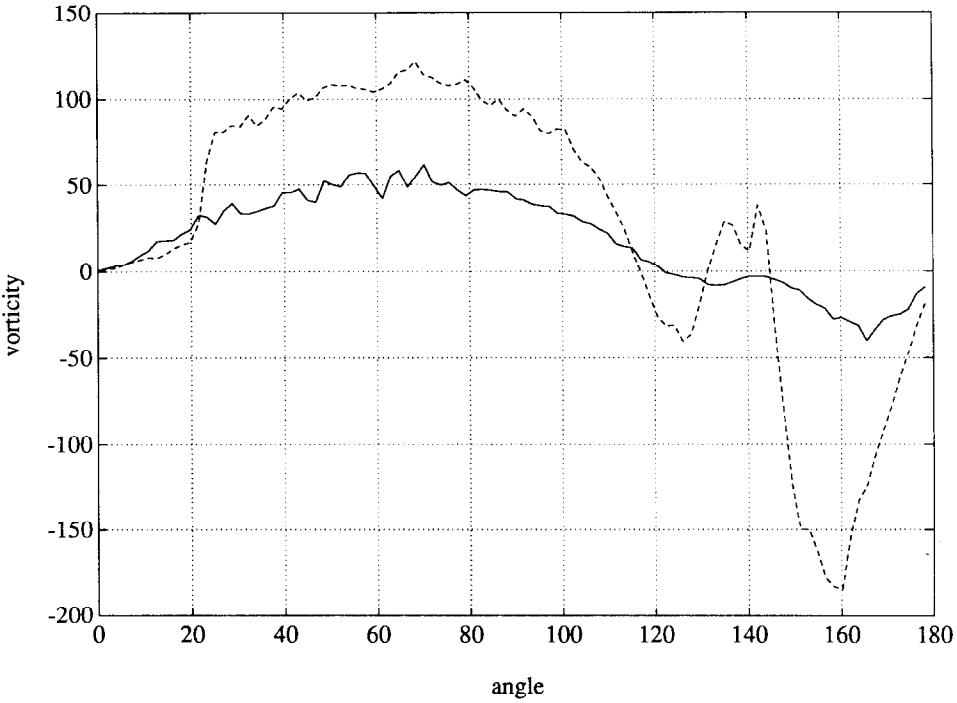


FIG. 7. Vorticity values on the surface of the cylinder at $t = 4$ for $\mathrm{Re} = 9500$ (dotted line) and $\mathrm{Re} = 3000$.

present in the rear part of the body, as opposed to what happens for lower Reynolds numbers. However, when compared to the numerical results in [9], these results attest to a lack of accuracy in the representation of the vorticity. This fact meets the classical observation that vortex methods, while giving accurate results for the velocity, suffer accuracy difficulties for the vorticity itself. Also one must keep in mind, when comparing with the results in [9], that in this reference the space resolution is significantly higher.

Figure 8 shows the velocity profile behind the cylinder at various times for $\mathrm{Re} = 3000$. Negative values correspond to a back flow which can be greater in absolute value than the velocity at infinity. However, experiments show that when the Reynolds number increases past the value of 1000, the strength of this back flow, as well as the wake length decrease. This is confirmed by Fig. 9, which is the analogue of Fig. 8 for $\mathrm{Re} = 9500$. In both Figs. 8 and 9 our numerical results are compared with experimental data as reported in [2]. The agreement is satisfactory with, however, a discrepancy appearing at large times for $\mathrm{Re} = 9500$.

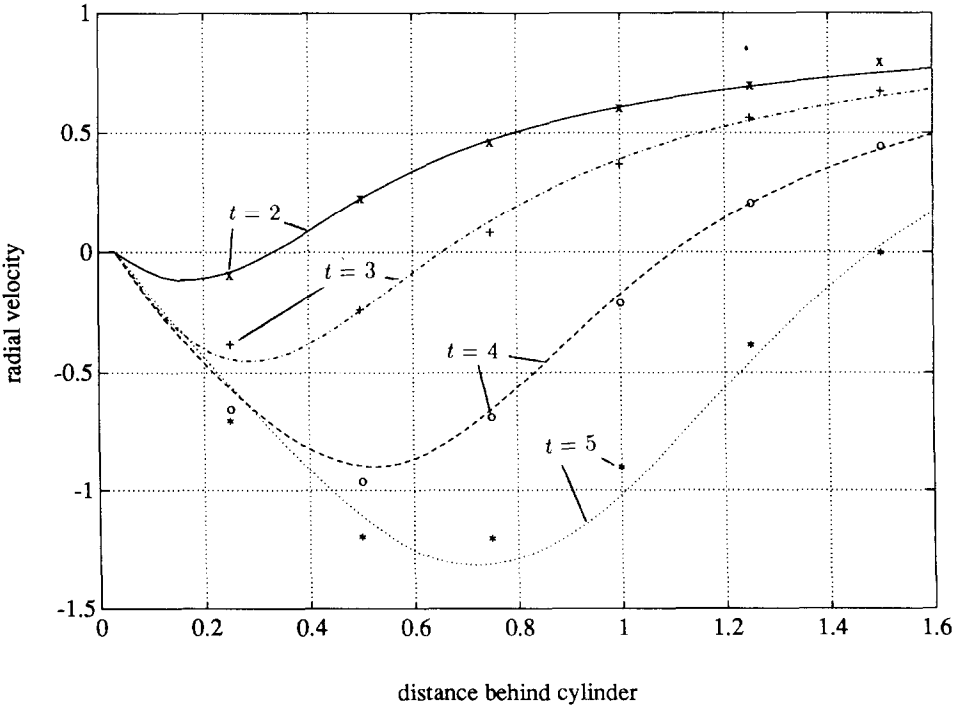


FIG. 8. Velocity profile on the axis behind the cylinder for $\mathrm{Re} = 3000$ at times 2, 3, 4, 5 (lines) compared with experimental results (points).

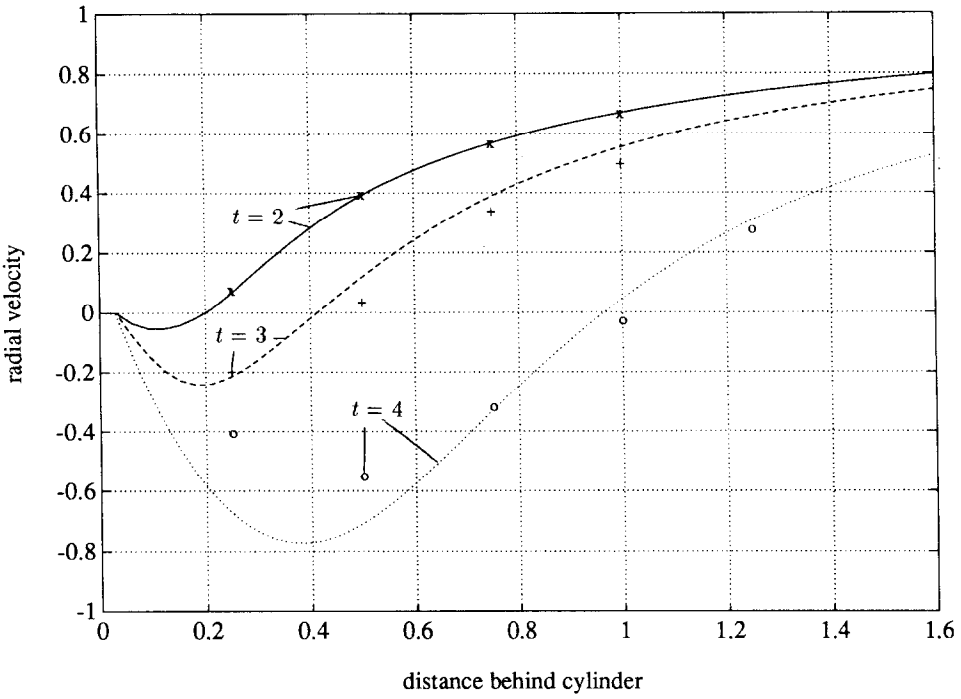


FIG. 9. Velocity profile on the axis behind the cylinder for Re = 9500 at times 2, 3, 4 (lines) compared with experimental results (points).

## 5. CONCLUSION

A particle-grid superposition method for solving the Navier–Stokes equations has been presented, whose philosophy is totally different from the conventional particle-in-cell methods. The idea is to maintain the advantages of particle methods in convection dominated flows while correcting the particle diffusion formula in the boundary layer. Given, on the one hand, the roughness of the artificial boundary conditions and of the polar grid used to compute the velocity and, on the other hand, the limited number of particles used in our simulations, the numerical results concerning the velocity profiles for small times can be considered as satisfactory. However some problems remain. Part of them are related to the fact that particles are still used in the boundary layer. In particular, the efficiency of the vorticity creation process seems to be very sensitive to the position of the first layer of particles. For such reasons we believe that the ultimate form of the proposed method would be a domain decomposition algorithm where a complete Navier–Stokes finite difference scheme would be used in the boundary layer. This would allow us to get rid of the particles in the very neighbourhood of the obstacle. Therefore dealing

with the boundary condition would be easier and the representation of the vorticity would presumably be more accurate in the boundary layer. In this situation particles would only need to be generated upstream in such a way that they are present at the interface between the two domains, and the key point would be to use (16) and (17) to ensure a correct transfer of vorticity between the boundary layer and the wake. Here we want to emphasize the fact that this matching procedure is strongly based on our choice of a deterministic approach for the vortex resolution of the diffusion, and it would not be appropriate with a random walk method. As a result it is different from the one proposed in [11] in several respects. One of them is the fact that in this later method, which combines random walk and finite difference schemes, the transfer of vorticity is achieved partly by means of particle generation at the interface, which requires some caution in order to prevent increasing the number of particles too much. To finish with further developments of the method we must also mention that the vortex-in-cell approach used here for computing the velocity of the lagrangian particles makes it necessary to use a grid. Although it has been found that a crude grid did not deteriorate the overall results for the velocity, for higher Reynolds numbers this grid could lead to a limitation. Also related to this grid, the necessity of introducing an artificial boundary condition for the computation of the velocity is an obstacle if one wishes to consider large time computations, as the wake is clearly artificially constrained to remain within the limits of the computational domain (we believe that this is the leading cause of discrepancy in the result in Fig. 9 for $t=4$). Therefore an other future improvement would be to replace this vortex-in-cell method by a fast vortex blob method, and in the future we plan to incorporate these ideas in a more extensive study of the wake at high Reynolds numbers.

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